Mean-field stochastic differential equations driven by G-Brownian motion

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Introduction

- Distribution dependent SDEs/ McKean-Vlasov SDE/ Mean-field SDEs
- $$\mathrm{d}X_t = b(t, X_t, L_{X_t})\mathrm{d}t + \sigma(t, X_t, L_{X_t})\mathrm{d}B_t,$$

  where the drift and diffusion coefficients depend not only on the state variable X_t, but also on its marginal distribution L_{X_t}.

Introduction

A brief review on McKean-Vlasov SDEs

- McKean 1966, Proc. Natl. Acad. Sci. USA.
- Vlasov 1968, Sov. Phys. Usp.
- Sznitman 1991, Topics in propagation of chaos.
- F.-Y. Wang 2018, Stochastic Process. Appl.
- J. Shao, j. Bao, C.Yuan, X. Huang, P. Ren

Introduction

McKean-Vlasov SDEs under G-expectation framework

- S.Q. Sun 2020, Math. Methods Appl. Sci.
- D. Sun; J.L. Wu; P.Y. Wu 2023, arXiv:2302.12539

G-Expectation

- Fix two positive constants $\underline{\sigma}$ and $\bar{\sigma}$ with $\underline{\sigma} < \bar{\sigma}$, define

$$G(A) = \frac{1}{2} \sup_{\gamma \in \mathbb{S}_+^m \bigcap [\underline{\sigma}^2 \mathbf{I}_{m\times m}, \bar{\sigma}^2 \mathbf{I}_{m\times m}]} \mathrm{tr}\,[A\gamma]\,, \;\; A \in \mathbb{S}^m.$$

- induced $\bar{\mathbb{E}}$; $\Omega_T = C_0([0,T];\mathbb{R}^d)$; $\omega_0 = 0$; $\|\cdot\|_\infty$; $B_t(\omega) = \omega_t$;

$$L_G^p(\Omega_T) := \{X \in L^0(\Omega_T) \mid \lim_{N\to\infty} \bar{\mathbb{E}}[|X|^p 1_{|X|\geq N}] = 0, \;\; X \text{ q.c.}\}.$$

- $M_G^p([0,T])$; $M_G^{p,0}([0,T])$, $\|\eta\|_{M_G^p([0,T])} := \left[\bar{\mathbb{E}}\left(\int_0^T |\eta_t|^p dt\right)\right]^{\frac{1}{p}}$,

$$M_G^{p,0}([0,T]) = \left\{\eta_t = \sum_{j=0}^{N-1} \xi_j 1_{[t_j,t_{j+1})}; \xi_j \in L_G^p(\Omega_{t_j})\right\}.$$

Mean-Field G-SDE

Consider

$$X_t = x_0 + \int_0^t b(s, X_s, \bar{\mathbb{E}} X_s) \mathrm{d}s + \int_0^t \sum_{i,j=1}^m h_{ij}(s, X_s, \bar{\mathbb{E}} X_s) \mathrm{d} \left\langle B^i, B^j \right\rangle_s + \int_0^t \left\langle \sigma(s, X_s, \bar{\mathbb{E}} X_s), \mathrm{d}B_s \right\rangle, \quad (1)$$

where $B_\cdot$ is an m-dimensional G-Brownian motion under $\bar{\mathbb{E}}$, and $\left\langle B^i, B^j \right\rangle_t$ stands for the mutual variation process of the i-th component B^i_t and the j-th component B^j_t, $\bar{\mathbb{E}} X_t$ is the expectation of X_t under $\bar{\mathbb{E}}$,

$$b, h_{ij} = h_{ji} : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n; \ \sigma : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times m}.$$

Mean-Field G-SDE

Assumptions

Assume the following two conditions are satisfied:

(A1) For fixed $x\ x' \in \mathbb{R}^n$,

$$b(\cdot, x, x'),\ \ h_{ij}(\cdot, x, x') \in L^2([0,T];\mathbb{R}^n),$$
$$\sigma(\cdot, x, x') \in L^2([0,T];\mathbb{R}^{n\times m}).$$

(A2) For $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$, b, h_{ij}, σ are satisfying the Lipschitz condition:

$$|b(s,x_1,y_1) - b(s,x_2,y_2)| + \sum_{i,j=1}^{m} |h_{ij}(s,x_1,y_1) - h_{ij}(s,x_2,y_2)|$$
$$+ |\sigma(s,x_1,y_1) - \sigma(s,x_2,y_2)| \le K\left(|x_1 - x_2| + |y_1 - y_2|\right),$$

Mean-Field G-SDE

Theorem 1

Assume **(A1)-(A2)**. Then the mean-field G-SDE (1) has a unique solution $X \in M_G^2([0, T]; \mathbb{R}^n)$.

- idea of a proof
  - Define a mapping:

$$\varphi : M_G^2\left([0, T]; \mathbb{R}^n\right) \to M_G^2\left([0, T]; \mathbb{R}^n\right)$$
$$x' \to X.$$

  - φ is a contraction mapping on $M_G^2([0, T]; \mathbb{R}^n)$.

Mean-Field G-SDE

Theorem 2

Let X_t be the solution of the mean-field G-SDE (1). Then

$$\bar{\mathbb{E}}\big(\sup_{t\in[0,T]}\big|X_t^x - X_t^y\big|^2\big) \leq \delta^2|x-y|^2,$$

where

$$\delta^2 = 16K^2(T + T\bar{\sigma}^2 + 4\bar{\sigma}^2). \tag{2}$$

Wang's inequalities

- Wang's inequalities

$$\Phi(Pf(x)) \le P\Phi(f)(y)\mathrm{e}^{\Psi(x,y)}, \; x, y \in \mathbb{R}^d, f \in \mathbf{B}_b^+(\mathbb{R}^d),$$

  - $P : \mathbf{B}_b^+(\mathbb{R}^d) \to \mathbf{B}_b^+(\mathbb{R}^d)$ linear operator;
  - $\Phi : [0, \infty) \to (0, \infty)$, convex;
  - $\Psi : \mathbb{R}^d \times \mathbb{R}^d \to (0, \infty)$.

•F.-Y. Wang (1997), PTRF.

•F.-Y. Wang (2013), Springer, New York.

Harnack and log-Harnack inequalities

- Mean-field G-SDE:

$$\mathrm{d}X_t = b(t, X_t, \bar{\mathbb{E}}X_t)\mathrm{d}t + h(t, X_t, \bar{\mathbb{E}}X_t)\mathrm{d}\langle B\rangle_t + \mathrm{d}B_t. \qquad (3)$$

- b and h satisfy assumptions **(A1)** and **(A2)**.
- Well-posedness
- Nonlinear operator

$$\bar{P}_T f(x) = \bar{\mathbb{E}}f(X_T^x),\ f \in C_b^+(\mathbb{R}),$$

$$L_G^p(\Omega_T) = \{X \in L^0(\Omega_T) \mid \lim_{N\to\infty} \bar{\mathbb{E}}[|X|^p 1_{|X|\geq N}] = 0,\ X\ q.c.\}.$$

Harnack and log-Harnack inequalities

Theorem 3

Under **(A1)** and **(A2)**, for any nonnegative $f \in C_b^+(\mathbb{R})$, $p > 1$ and $T > 0, x, y \in \mathbb{R}$, it holds that

(1) Harnack inequality

$$(\bar{P}_T f)^p(y) \le \bar{P}_T f^p(x) \exp\left\{ \frac{p}{2(p-1)} \Lambda(\underline{\sigma}, \bar{\sigma}, T, K)|x-y|^2 \right\} \quad (4)$$

(2) Log-Harnack inequality

$$\bar{P}_T \log f(y) \le \log \bar{P}_T f(x) + \frac{1}{2}\Lambda(\underline{\sigma}, \bar{\sigma}, T, K)|x-y|^2.$$

where $\Lambda(\underline{\sigma}, \bar{\sigma}, T, K) :=$
$\left(\frac{\underline{\sigma}^{-2}}{T} + (1+\underline{\sigma}^{-2})(1+2\delta)K + \frac{(\underline{\sigma}^{-2}+\bar{\sigma}^2+2)(1+3\delta(1+\delta))K^2 T}{3} \right)$.

Harnack and log-Harnack inequalities

Proofs

Let $\{X_t^x\}_{t\geq 0}$ solve (3) with $X_0 = x$. Define $\mu_t := \bar{\mathbb{E}} X_t^x$,

$$\mathrm{d}X_t^x = b(t, X_t^x, \mu_t)\mathrm{d}t + h(t, X_t^x, \mu_t)\mathrm{d}\langle B\rangle_t + \mathrm{d}B_t.$$

Consider the following coupled stochastic differential equations

$$\begin{aligned}
\mathrm{d}X_t &= b(t, X_t, \mu_t)\mathrm{d}t + h(t, X_t, \mu_t)\mathrm{d}\langle B\rangle_t + \mathrm{d}B_t, \ X_0 = x = \mu_0, \\
\mathrm{d}Y_t &= b(t, Y_t, \nu_t)\mathrm{d}t + h(t, Y_t, \nu_t)\mathrm{d}\langle B\rangle_t + \mathrm{d}B_t \\
&\quad + \left(b(t, X_t, \mu_t) - b(t, Y_t, \nu_t) - \frac{v}{T}\right)\mathrm{d}t \\
&\quad\quad + \big(h(t, X_t, \mu_t) - h(t, Y_t, \nu_t)\big)\mathrm{d}\langle B\rangle_t, \ Y_0 = y = x + v = \nu_0,
\end{aligned}$$

where $\nu_t = \bar{\mathbb{E}} X_t^y$.
Note that $X_T = Y_T$.

Harnack and log-Harnack inequalities

Proofs

Let

$$u_t = b(t, X_t, \mu_t) - b(t, Y_t, \nu_t) - \frac{v}{T}, \; w_t = h(t, X_t, \mu_t) - h(t, Y_t, \nu_t).$$

From the condition **(A2)**, we obtain

$$\begin{aligned} |u_t| &= \left| \left(b(t, X_t, \mu_t) - b(t, Y_t, \nu_t) - \frac{v}{T} \right) \right| \\ &\leq K \left(|X_t - Y_t| + |\mu_t - \nu_t| \right) + \frac{|v|}{T}, \\ |w_t| &= |(h(t, X_t, \mu_t) - h(t, Y_t, \nu_t))| \\ &\leq K \left(|X_t - Y_t| + |\mu_t - \nu_t| \right). \end{aligned}$$

Harnack and log-Harnack inequalities

Proofs

It follows from Theorem 2 that

$$|\mu_t - \nu_t| = \left|\bar{\mathbb{E}}X_t^x - \bar{\mathbb{E}}X_t^y\right| \leq \bar{\mathbb{E}}\left(\left|X_t^x - X_t^y\right|\right) \leq \delta|x-y|.$$

Therefore

$$\begin{aligned} |u_t| &\leq K\left(\left|\frac{t-T}{T}v\right| + \delta|x-y|\right) + \frac{|v|}{T} \leq \frac{1+K((1+\delta)T-t)}{T}|v|, \\ |w_t| &\leq K\left(\left|\frac{t-T}{T}v\right| + \delta|x-y|\right) \leq \frac{K((1+\delta)T-t)}{T}|v|. \end{aligned} \tag{5}$$

Harnack and log-Harnack inequalities

- Hu et al. $(\widetilde{\Omega}_T, L^1_{\widetilde{G}}, \bar{\mathbb{E}}^{\widetilde{G}})$, $\widetilde{\Omega}_T = C_0([0,T];\mathbb{R}^{2m})$
- $\widetilde{G}(A) = \frac{1}{2}\sup_{v\in[\underline{\sigma}^2\mathbf{I}_{m\times m},\bar{\sigma}^2\mathbf{I}_{m\times m}]} \mathrm{tr}\left[A\begin{pmatrix} v & \mathbf{I}_{m\times m} \\ \mathbf{I}_{m\times m} & v^{-1}\end{pmatrix}\right].$
- $A\in\mathbb{S}^{2m}$.
- Let $(B_t,\bar{B}_t)$ be the canonical process in the extended space.
- $\langle B_t,\bar{B}_t\rangle = t\mathbf{I}_{m\times m}$, $\bar{\mathbb{E}}^{\widetilde{G}}[\xi]=\bar{\mathbb{E}}[\xi]$, $\xi\in L^1_G(\Omega_T)$, $L_{B|_{\bar{\mathbb{E}}}} = L_{B|_{\bar{\mathbb{E}}^{\widetilde{G}}}}$.
- $\bar{B}_t$ is a $\hat{G}$-Brownian motion under $\bar{\mathbb{E}}^{\widetilde{G}}$ with

$$\hat{G}(A) = \frac{1}{2}\sup_{\bar{\sigma}^{-2}\mathbf{I}_{m\times m}\le v\le\underline{\sigma}^{-2}\mathbf{I}_{m\times m}} \mathrm{trace}[Av], A\in\mathbb{S}^m.$$

- $\bar{\mathbb{E}}[f(X_T^x)] = \bar{\mathbb{E}}^{\widetilde{G}}[f(X_T^x)] =: \bar{P}_T^{\widetilde{G}}f(x)$.

Proofs

引理

Let $(f_t)_{t\le T}$, $(g_t)_{t\le T} \in M_G^2([0,T];\mathbb{R}^d)$. *If*

$$\bar{\mathbb{E}}^{\widetilde{G}} \exp\left\{\left(\frac{1}{2}+\delta\right)\int_0^T \left(\langle f_s, \mathrm{d}\langle \bar{B}\rangle_s f_s\rangle + \langle g_s, \mathrm{d}\langle B\rangle_s g_s\rangle + 2\langle f_s, g_s\rangle \mathrm{d}s\right)\right\} < \infty,$$

$\delta > 0$ *is a consant, then the process* $\widehat{B} := B + \int_0^\cdot f_s \mathrm{d}s + \int_0^\cdot g_s \mathrm{d}\langle B\rangle_s$ *is a* G*-Brownian motion on* $[0,T]$ *under* $\widehat{\mathbb{E}}[\cdot] := \bar{\mathbb{E}}^{\widetilde{G}}[R_T(\cdot)]$ *with*

$$\begin{aligned} R_T = \exp\Bigg\{ & -\int_0^T \left\langle \begin{pmatrix} f_s \\ g_s \end{pmatrix}, \mathrm{d}\begin{pmatrix} \bar{B}_s \\ B_s \end{pmatrix} \right\rangle \\ & - \frac{1}{2}\int_0^T \left(\langle f_s, \mathrm{d}\langle \bar{B}\rangle_s f_s\rangle + \langle g_s, \mathrm{d}\langle B\rangle_s g_s\rangle + 2\langle f_s, g_s\rangle \mathrm{d}s\right)\Bigg\}. \end{aligned}$$

Proofs

$$
\begin{aligned}
M_T := \exp\Bigg\{ & -\int_0^T \left\langle \begin{pmatrix} u_s \\ w_s \end{pmatrix}, \mathrm{d} \begin{pmatrix} \bar{B}_s \\ B_s \end{pmatrix} \right\rangle \\
& - \frac{1}{2}\int_0^T \left(\left\langle u_s, \mathrm{d}\left\langle \bar{B} \right\rangle_s u_s \right\rangle + \left\langle w_s, \mathrm{d}\left\langle B \right\rangle_s w_s \right\rangle + 2\left\langle u_s, w_s \right\rangle \mathrm{d}s \right) \Bigg\} \\
= \exp\Bigg\{ & -\int_0^T w_s \mathrm{d}B_s - \frac{1}{2}\int_0^T |w_s|^2 \mathrm{d}\langle B\rangle_s - \int_0^T w_s u_s \mathrm{d}s \\
& - \int_0^T u_s \mathrm{d}\bar{B}_s - \frac{1}{2}\int_0^T |u_s|^2 \mathrm{d}\langle \bar{B}\rangle_s \Bigg\}.
\end{aligned}
$$

Proofs

Define a sublinear expectation $\widehat{\mathbb{E}}$ by $\widehat{\mathbb{E}}[\xi] := \bar{\mathbb{E}}^{\widetilde{G}}[\xi M_T]$, then the process

$$\widehat{B}_t := B_t + \int_0^t u_s \mathrm{d}s + \int_0^t w_s \mathrm{d}\langle B\rangle_s, \quad t \geq 0$$

is a G-Brownian motion under $\widehat{\mathbb{E}}$. Then, Y_t can be expressed by

$$\begin{aligned} \mathrm{d}Y_t &= b(t, Y_t, \nu_t)\mathrm{d}t + h(t, Y_t, \nu_t)\mathrm{d}\langle B\rangle_t + \mathrm{d}B_t + u_t \mathrm{d}t + w_t \mathrm{d}\langle B\rangle_t \\ &= b(t, Y_t, \nu_t)\mathrm{d}t + h(t, Y_t, \nu_t)\mathrm{d}\langle \widehat{B}\rangle_t + \mathrm{d}\widehat{B}_t. \end{aligned}$$

Proofs

Now we come to derive the Harnack inequality as follows

$$\bar{P}_T f(y) = \bar{\mathbb{E}} f(X_T^y) = \widehat{\mathbb{E}} f(Y_T^y) = \widehat{\mathbb{E}} f(X_T^x) = \bar{\mathbb{E}}^{\widetilde{G}}(M_T f(X_T^x)). \quad (6)$$

By Hölder's inequality, we obtain

$$(\bar{P}_T f)^p(y) = (\bar{\mathbb{E}}^{\widetilde{G}}[M_T f(X_T^x)])^p \leq (\bar{\mathbb{E}}^{\widetilde{G}}[f^p(X_T^x)]) \left(\bar{\mathbb{E}}^{\widetilde{G}} \left[M_T^{\frac{p}{p-1}} \right] \right)^{p-1}. \quad (7)$$

Proofs

Moreover,

$$\begin{aligned}\bar{\mathbb{E}}^{\widetilde{G}}\left[M_T^{\frac{p}{p-1}}\right] =&\bar{\mathbb{E}}^{\widetilde{G}}\exp\Bigg\{-\frac{p}{p-1}\int_0^T u_s\mathrm{d}\bar{B}_s-\frac{p}{2(p-1)}\int_0^T|u_s|^2\mathrm{d}\langle\bar{B}\rangle_s\\&-\frac{p}{p-1}\int_0^T u_s w_s\mathrm{d}s-\frac{p}{p-1}\int_0^T w_s\mathrm{d}B_s\\&-\frac{p}{2(p-1)}\int_0^T|w_s|^2\mathrm{d}\langle B\rangle_s\Bigg\}\\=&\bar{\mathbb{E}}^{\widetilde{G}}\exp\Bigg[\frac{p}{2(p-1)^2}\Big(\int_0^T|u_s|^2\mathrm{d}\langle\bar{B}\rangle_s+\int_0^T|w_s|^2\mathrm{d}\langle B\rangle_s\\&+\int_0^T 2u_s w_s\mathrm{d}s\Big)\Bigg].\end{aligned} \tag{8}$$

Proofs

Combining with (5), we deduce that

$$
\begin{aligned}
&\int_0^T |u_s|^2 \mathrm{d}\langle \bar{B}\rangle_s + \int_0^T |w_s|^2 \mathrm{d}\langle B\rangle_s + \int_0^T 2u_s w_s \mathrm{d}s \\
&\leq \underline{\sigma}^{-2}\int_0^T |u_s|^2 \mathrm{d}s + \bar{\sigma}^2 \int_0^T |w_s|^2 \mathrm{d}s + \int_0^T 2u_s w_s \mathrm{d}s \\
&\leq \Lambda(\underline{\sigma}, \bar{\sigma}, T, K)|x-y|^2,
\end{aligned} \tag{9}
$$

where $\Lambda(\underline{\sigma}, \bar{\sigma}, T, K) = \frac{\underline{\sigma}^{-2}}{T} + (1+\underline{\sigma}^{-2})(1+2\delta)K + \frac{\left(\underline{\sigma}^{-2}+\bar{\sigma}^2+2\right)(1+3\delta(1+\delta))K^2T}{3}$.

Harnack and log-Harnack inequalities

Proofs (1)

Combining this with (8), we have

$$\bar{\mathbb{E}}^{\widetilde{G}}\left[M_T^{\frac{p}{p-1}}\right] \leq \exp\left\{\frac{p}{2(p-1)^2}\Lambda(\underline{\sigma}, \bar{\sigma}, T, K)|x-y|^2\right\}.$$

Substituting this into (7), we prove (4).

Harnack and log-Harnack inequalities

Proofs (2)

For the log-Harnack inequality, similar to (6), we have

$$\begin{aligned}\bar{P}_T \log f(y) &= \bar{\mathbb{E}} \log f(X_T^y) = \widehat{\mathbb{E}} \log f(Y_T^y)\\ &= \widehat{\mathbb{E}} \log f(X_T^x) = \bar{\mathbb{E}}^{\widetilde{G}}(M_T \log f(X_T^x)).\end{aligned}$$

According to Young inequality, we obtain

$$\begin{aligned}\bar{\mathbb{E}}^{\widetilde{G}}(M_T \log f(X_T^x)) &\le \log \bar{\mathbb{E}}^{\widetilde{G}}[f(X_T^x)] + \bar{\mathbb{E}}^{\widetilde{G}}[M_T \log M_T]\\ &= \log \bar{P}_T f(x) + \bar{\mathbb{E}}^{\widetilde{G}}[M_T \log M_T]\\ &= \log \bar{P}_T f(x) + \widehat{\mathbb{E}}[\log M_T].\end{aligned} \tag{10}$$

Harnack and log-Harnack inequalities

Proofs (2)

Let

$$\widehat{\bar{B}}_t := \bar{B}_t + \int_0^t w_s \mathrm{d}s + \int_0^t u_s \mathrm{d}\langle \bar{B}\rangle_s.$$

$\widehat{\bar{B}}_t$ is a $\widehat{G}$-Brownian motion under $\widehat{\mathbb{E}}$. Then, we have

$$\begin{aligned}\widehat{\mathbb{E}}[\log M_T] &= \widehat{\mathbb{E}}\Big[\frac{1}{2}\int_0^T |w_s|^2 \mathrm{d}\langle B\rangle_s - \int_0^T u_s \left(\mathrm{d}\widehat{\bar{B}}_s - w_s \mathrm{d}s - u_s \mathrm{d}\langle \bar{B}\rangle_s\right) \\ &\quad - \frac{1}{2}\int_0^T |u_s|^2 \mathrm{d}\langle \bar{B}\rangle_s\Big] \\ &= \frac{1}{2}\widehat{\mathbb{E}}\left[\int_0^T |w_s|^2 \mathrm{d}\langle B\rangle_s + \int_0^T |u_s|^2 \mathrm{d}\langle \bar{B}\rangle_s + \int_0^T 2u_s w_s \mathrm{d}s\right].\end{aligned}$$

Harnack and log-Harnack inequalities

Proofs (2)

Combining this with (9), we deduce that

$$\widehat{\mathbb{E}}[\log M_T] \leq \Lambda(\underline{\sigma}, \bar{\sigma}, T, K)\frac{|x-y|^2}{2}. \tag{11}$$

Substituting (11) into the last equation of (10), the prove of (17) is done.

Thanks!